INTERPERSONAL INDEPENDENCE OF KNOWLEDGE AND BELIEF

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ABSTRACT. We show that knowledge satisfies *interpersonal independence*, meaning that a non-trivial sentence describing one agent's knowledge cannot be equivalent to a sentence describing another agent's knowledge. The same property of interpersonal independence holds, mutatis mutandis, for belief. In the case of knowledge, interpersonal independence is implied by the fact that there are no non-trivial sentences that are common knowledge in every model of knowledge. In the case of belief, interpersonal independence follows from a *strong interpersonal independence* that knowledge does not have. Specifically, there is no sentence describing the beliefs of one person that implies a sentence describing the beliefs of another person.

Keywords: Interpersonal independence, Strong interpersonal independence, Knowledge, Belief, Common knowledge, Partition.

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1. Introduction

Interactive epistemology. *Epistemic attitudes* toward sentences come in many shapes and forms: knowledge, qualitative belief, and quantitative belief, each of which further splits into a variety of attitudes. Since we are dealing here only with epistemic attitudes, we will call them attitudes for short. The subjects to which epistemic attitudes are attributed can vary and be, for example, individual human beings, firms, states, or computers. We refer to such subjects as *agents*.

Of particular interest is the study of attitudes in environments that involve multiple agents. This study provides insights into how agents interact, make decisions, and cooperate or compete with each other. The study of such environments is relevant to a range of disciplines, including psychology, economics, game theory, computer science, sociology, biology, and political science.

The epistemic attitude of each agent in an interactive environment refers not only to objective facts, but also to subjective facts, namely, the attitudes of other agents. Furthermore, the attitudes of agents may also concern attitudes that refer to attitudes, and so on. For multi-agent environments, it is of interest to consider the extent to which one agent's attitude is independent of another agent's attitude. We study this question with respect to two commonly used versions of knowledge and (qualitative) belief.

We express the idea that the attitudes of two agents are independent in terms of the relation between *descriptions* of the agents' attitudes. We first explain what these descriptions are via several instances. Consider for example the following description of Alice's knowledge.

> (1) Alice knows that the black horse won the race.

Descriptions may include several simple descriptions, such as:

> (2) Either Alice knows that the black horse won the race, or Alice does not know that the red horse won the race.

Description of an agent's knowledge may include knowledge of the agent about another agent's knowledge, like the following:

> (3) Alice knows that Bob does not know that the black horse won the race.

If we replace 'Alice knows' with 'Alice believes' in these examples we get descriptions of Alice's belief. We can also replace 'Alice knows' with 'Alice ascribes probability p' to obtain descriptions of Alice's probabilistic beliefs.

Interpersonal independence. We define two conditions of independence in terms of the relations between the descriptions of the attitudes of two agents.

> *An attitude satisfies* interpersonal independence *if no description of the attitude of one agent is equivalent to the description of the attitude of another agent, except for trivial cases.*[1]

In other words, an attitude does *not* satisfy interpersonal independence if there are non-trivial descriptions of both Alice's and Bob's attitudes such that one description is true, if and only if the other is true.

We can require more from independence. Not only that no two descriptions can be equivalent, but also that no description of one agent's attitude tells us anything about the attitude of another agent. This leads us to the following definition.

> *An attitude satisfies* strong interpersonal independence *if no description of the attitude of one agent implies a description of the attitude of another agent, except for trivial cases*

We now check which of the two of independence conditions formulated above are satisfied by the attitudes of knowledge and belief. Beginning with the condition of strong interpersonal independence, we find that:

> *Belief satisfies strong interpersonal independence, but knowledge does not.*

To see why knowledge does not satisfy strong interpersonal independence, consider the description of Alice's knowledge in (1). Now, knowledge of a sentence implies the sentence. That is, if it is true that the sentence is known, then the sentence is true. Thus, (1) implies that the black horse won the race. But this implies in turn that Bob cannot know the opposite, that is:

> (4) Bob does not know that the black horse did *not* win the race.

In summary, the description of Alice's knowledge in (1) implies the description of Bob's knowledge in (4).

The descriptions of belief obtained from (1) and (4) by changing 'know' to 'believe' do not have this relation of implication, because belief in a sentence does not imply the sentence. Indeed, consider the following two descriptions of belief:

> (5) Alice believes that the black horse won the race.
> (6) Bob believes that the black horse did not win the race.

Obviously, one and only one of these two believed sentences is true. Yet (5) and (6) can be both true, and therefore (5) does not imply the negation of (6). This does not yet prove that belief satisfies strong interpersonal independence because we need to show that no description of Alice's belief, no matter how complicated, does not imply any description of Bob's belief. We show that this is indeed the case.

[1]By trivial cases we mean descriptions that are either true in every model of knowledge, or false in every model. Equivalently, by the completeness theorem, the trivial cases are either theorems or contradictions.

A straightforward corollary is:

Belief satisfies interpersonal independence.

As knowledge does not satisfy strong interpersonal independence, the question of whether it satisfies interpersonal independence cannot be answered so easily. Our main result shows, however, that knowledge does satisfy interpersonal independence. We can summarize our results:

Both knowledge and belief satisfy interpersonal independence. Only belief satisfies strong interpersonal independence.

The formalism. We study knowledge and belief as modalities in a formal language of modal logic as was first suggested by Hintikka (1962). For belief we use the modal logic KD45.[2] We discuss later the relation between this type of belief and probabilistic belief. Knowledge is modeled here by the modal logic S5. This type of logic is obtained by adding to the axioms KD45 an axiom known as the *truth axiom*, which says that knowledge of a sentence implies the sentence. Some philosophical reservations concerning the rendering of knowledge in terms of the S5 logic were raised by Hintikka (1962) and Stalnaker (2006). Nevertheless, this model of knowledge is by far the most commonly used by practitioners in the sciences that study interactive epistemology.

Kripke (1963) proposed semantics for various modal logics in terms of models called Kripke structures, which consist of a set of possible worlds and accessibility relation between them. Each sentence in the formal language is associated in each model with a subset of worlds, considered to be the interpretation of the sentence in the model.

In the language of multi-agent modal logics there is one modality for each agent. Similarly, the semantics for such logics has an accessibility relation for each agent. Aumann (1976) studied multi-agent knowledge using partition models. In such a model each agent is associated with a partition of a state space that describes the agent's knowledge: in each state the agent knows all the supersets of the partition element that contains the state. Partition models are equivalent to Kripke structures for the S5 logic.

Common knowledge. To establish that knowledge satisfies interpersonal independence, we use the concept of common knowledge in a partition model, as defined in Aumann (1976). Common knowledge is defined by the *meet* partition, which is the finest common coarsening of the agents' partitions. A *common knowledge event* is any element of the field generated by the meet partition.[3] A sentence is *common knowledge* in a partition model if it is interpreted as a common knowledge event. The claim that knowledge satisfies interpersonal independence is equivalent to the following claim:

[2]KD45 is the acronym of the names of the four axioms that define the logic.

[3]Including the empty set, which we consider a common knowledge event for convenience.

There are no sentences that are common knowledge in every model except for the trivial ones.[4]

To show this we construct for any non-trivial sentence a model in which (i) the meet contains only one set, namely the whole state space, and (ii) this sentence and its negation are interpreted as nonempty events. Since the meet, in this case, does not contain any proper subset that is a common knowledge event, it follows that the sentence is not interpreted in this model as a common knowledge event.

The lack of no non-trivial sentences that are common knowledge in every model reveals a gap between the syntax and the semantics of S5 knowledge. While common knowledge is well defined in every model of knowledge, it is impossible to define common knowledge syntactically in terms of the agents' knowledge. Halpern, Samet and Segev (2009) discuss the definability of a modality in terms of other modalities. Halpern, Samet and Segev (2009) demonstrates another gap between syntax and semantics. They show that S5 knowledge cannot be syntactically defined in terms KD45 belief, while in every model of KD45 belief, S5 knowledge can be defined in a unique way. The reason for this gap is that Kripke structures have some features that are not shared by other models of modal logic. The syntax can reflect only properties that are shared by all possible models. Samet (2010) provides examples of models of possible worlds for S5 knowledge that cannot be described in terms of accessibility relation and therefore are not Kripke structures.

Probabilistic beliefs. The relation between discrete models of KD45 belief and discrete models of probabilistic belief is summarized succinctly in Samet (2013). In every discrete model of probabilistic belief, the restriction of the probabilistic belief to certainty, namely belief in probability 1, is a model of KD45 belief. Conversely, every discrete model of KD45 belief can be extended to a model of probabilistic belief. Halpern (1991) studies this relation in logics of probabilistic belief. However, non-discrete models of probabilistic belief require a measure-theoretic structure, while no such structure is required for non-discrete models of KD45 belief.

A universal space for probabilistic beliefs is constructed in Mertens, Zamir (1985). This space is the product of type spaces, one for each agent. The type space of an agent consists of all hierarchies of the agent's beliefs. Because of the product structure, each type of one agent is consistent with any type of another agent. While the construction of types in Mertens, Zamir (1985) requires the use of topology, Heifetz, Samet (1998) constructed the universal space using only measure-theoretic notions. Moreover, this space is constructed in syntactic terms using sentences rather than hierarchies of beliefs. Since a sentence that describes an agent's probabilistic beliefs corresponds to a set of types

[4]See footnote 1.

of this agent, the product structure of the universal space guarantees that probabilistic beliefs are interpersonal independent in the sense defined here. We cannot use directly this result to prove that KD45 belief is interpersonal independence since the construction of the universal space requires topology or measure theory while belief does not require it. The proof that KD45 belief is interpersonal independent is much simpler than the construction of the universal space.

2. Preliminaries

2.1. **The syntax of the logic of knowledge.** We consider a logic of multi-agent knowledge for a finite set I of individuals.[5] The set of *sentences* $\mathcal{S}^K$ of this logic is defined, starting with a set A of *atomic sentences*, using *propositional connectors*, $\neg$, $\rightarrow$, $\wedge$, and $\vee$, and knowledge operators K_i. Formally: (i) every atomic sentence is a sentence; (ii) if φ and ψ are sentences, then $\neg\varphi$ (read, not φ), $(\varphi \rightarrow \psi)$ (read, if φ then ψ), $(\varphi \wedge \psi)$ (read, φ and ψ), $(\varphi \vee \psi)$ (read, φ or ψ), are sentences; (iii) if φ is a sentence, then for each $i \in I$, $K_i\varphi$ (read, i knows φ) is a sentence. The set of sentences $\mathcal{S}^k$ is the smallest set that satisfies (i), (ii), and (iii). We denote by $\mathcal{S}^k_i$ the set of sentences that *describe i's knowledge*, that is, the sentences that are generated from the set $\{K_i\varphi \mid \varphi \in \mathcal{S}^k\}$ by the propositional connectors.

The subset of *theorems* in $\mathcal{S}^K$ is defined inductively starting with a set of sentences called *axioms*. The set of axioms consists of all propositional calculus tautologies and for any i and any tow sentences φ and ψ, each of the following sentences:

(K) $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$;
(T) $K_i\varphi \rightarrow \varphi$ (truth axiom);
(5) $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$ (negative introspection).

The set of theorems is the smallest set of sentences that satisfies the following three properties: (1) each axiom is a theorem; (2) if φ and $\varphi \rightarrow \psi$ are theorems then ψ is a theorem (*modus ponens*); (3) if φ is a theorem, then for any i, $K_i\varphi$ is a theorem (*generalization*). The negation of a theorem is called a *contradiction*. A sentence that is not a contradiction is *consistent*. When $\varphi \rightarrow \psi$ is a theorem we say that φ *logically implies* ψ and write $\varphi \Rightarrow \psi$. If φ logically implies ψ and vice versa we say that φ and ψ are *logically equivalent* and write $\varphi \Leftrightarrow \psi$.

2.2. **The semantics of knowledge.** We use *Kripke models*, here *models* for short, as the semantics of the logic. A model is a tuple $M = (\Omega, (\rightsquigarrow_i)_{i\in I}, [\,\cdot\,])$, where

- Ω is a set of elements called *worlds* or *states*;
- For each i, $\rightsquigarrow_i$ is a reflexive, symmetric, and transitive binary relation on Ω, called *accessibility* relation;

[5]This logic is described in detail in Fagin, Halpern, Moses and Vardi (1995).

- $[\,\cdot\,]\colon A \to 2^{\Omega}$ is an *interpretation* function that assigns to each atomic sentences a subset of Ω.

The interpretation of atomic sentences in the model is extended to all sentences. Thus, for each sentence φ we defined a subset of Ω denoted by $[\varphi]$. The interpretation is defined inductively; if $[\varphi]$ and $[\psi]$ are defined, then $[\neg\varphi] = \Omega \setminus [\varphi]$; $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$; $[\varphi \vee \psi] = [\varphi] \cup [\psi]$; $[\varphi \to \psi] = (\Omega \setminus [\varphi]) \cup [\psi]$; and $[K_i\varphi] = \{\omega \mid \text{if } \omega \to_i \omega',\ \text{then } \omega' \in [\varphi]\}$. That is, a world ω is in $[K_i\varphi]$ if φ is true at all the worlds accessed from ω. When $\omega \in [\varphi]$ in the model M we say that φ *is true in* ωin M and write $M, \omega \models \varphi$.

The *completeness theorem* states that for any sentence φ, φ is a theorem if and only if in each model, φ is true in all the worlds of the model, that is, $[\varphi] = \Omega$. It follows that φ is a contradiction if and only if in every model $[\varphi] = \emptyset$. Also, φ is consistent if and only if there exists a model and a world in the model where φ is true. By the definition of interpretation and the completeness theorem, $\varphi \Rightarrow \psi$ if and only if $[\varphi] \subseteq [\psi]$ in every model, and $\varphi \Leftrightarrow \psi$ if and only if $[\varphi] = [\psi]$ in every model.

2.3. **The logic of belief.** The set of sentences $\mathcal{S}^B$ of this logic is defined similarly to the set of sentences of the logic of knowledge, $\mathcal{S}^K$, except that the knowledge operators K_i are replaced by belief operators B_i. The set of axioms consists of all tautologies and for each i, φ, and ψ, any sentence in the following list:

(K) $B_i(\varphi \to \psi) \to (B_i\varphi \to B_i\psi)$;
(D) $B_i\varphi \to \neg B_i\neg\varphi$ (*consistency*);
(4) $B_i\varphi \to B_iB_i\varphi$ (*positive introspection*);
(5) $\neg B_i\varphi \to B_i\neg B_i\varphi$ (*negative introspection*).

The set of theorems in $\mathcal{S}^B$ is defined similarly to the set of theorems in $\mathcal{S}^K$, with the knowledge operators K_i replaced by belief operators B_i.

It is easy to see that (D) and (4) are theorems of the logic of knowledge. Thus, the only difference between knowledge and belief is that the former satisfies the truth axiom and the latter does not.

2.4. **The semantics of belief.** A model for the logic of belief is defined like a model of the logic of knowledge except that the properties of the accessibility relations are as follows: $\rightsquigarrow_i$ is transitive, serial (that is, for each ω there is an ω' such that $\omega \rightsquigarrow \omega'$), and Euclidean (that is, if $\omega \rightsquigarrow_i \omega'$ and $\omega \rightsquigarrow_i \omega''$, then $\omega' \rightsquigarrow_i \omega''$). It is readily seen that if such a relation is also reflexive, then it is an equivalence relation. The interpretation of sentences in a model of the logic of belief is defined like the interpretation in the case of the logic of knowledge, with B_i replacing K_i. The completeness theorem holds also for the logic of belief.

2.5. **Replacing accessibility by subsets.** There is a one-to-one mapping between accessibility relations $\rightsquigarrow_i$ that are equivalence relations and partitions of Ω. Each such relation $\rightsquigarrow_i$ is uniquely associated with a partition Π_i of Ω into equivalence classes. Define Π_i to be the partition of Ω into the equivalence classes of the accessibility relation. We denote by $\Pi_i(\omega)$ the element of Π_i that contains ω. We can now equivalently define the interpretation of $K_i\varphi$ is terms of Π_i: $[K_i\varphi] = \{\omega \mid \Pi_i(\omega) \subseteq [\varphi]\}$. In what follows we sometimes describe a model for the logic of knowledge as $M = (\Omega, (\Pi_i)_{i\in I}, [\,\cdot\,])$ where the elements of the partitions Π_i are the equivalence classes of $\rightsquigarrow_i$.

The definition of belief models in terms of subsets rather than accessibility relations is done as follows. There is a one-to-one mapping of accessibility relations $\rightsquigarrow_i$ that are transitive, serial, and Euclidean onto pairs (Π_i, Σ_i) with the following properties:

- Π_i is a partition of Ω;
- Σ_i is a family of non-empty sets, called *supports*, such that each element of Π_i contains exactly one support.

Starting with such a relation $\rightsquigarrow_i$, denote by S the subset of worlds in Ω that are accessible from some world. This set has a partition Σ_i such that for each $\sigma \in \Sigma_i$ all the worlds in σ are accessible from each other but do not accesses any world out of σ. For $\sigma \in \Sigma_i$ let $\pi(\sigma)$ be the set of worlds that access worlds in σ. Obviously, $\sigma \subseteq \pi(\sigma)$ and moreover, each world in $\pi(\sigma) \setminus \sigma$ accesses all the worlds in σ and only these worlds. The family of sets $\pi(\sigma)$ forms a partition Π_i of Ω. Starting with a pair (Π_i, Σ_i) as described above we can easily construct the unique transitive, serial, and Euclidean accessibility relation that gives rise to the pair. We denote by $\Pi_i(\omega)$ the element of Π_i that contains ω, and by $\Sigma_i(\omega)$ the element in Σ_i contained in $\Pi_i(\omega)$. We can equivalently redefine B_i in terms of Σ_i as $[B_i\varphi] = \{\omega \mid \Sigma_i(\omega) \subseteq [\varphi]\}$. In what follows we sometimes describe a model for the logic of belief as $M = (\Omega, (\Pi_i, \Sigma_i)_{i\in I}, [\,\cdot\,])$, where the elements of the partitions Π_i are the equivalence classes of $\rightsquigarrow_i$.

Claim 1. *If $\varphi \in \mathcal{S}_i^K$ ($\varphi \in \mathcal{S}_i^B$) then in every model, $[\varphi]$ is a union of elements of Π_i.*

By definition, for each sentence $K_i\varphi$, $[K_i\varphi]$ is a union of elements of Π_i. Moreover, if φ and ψ are sentences such that in each model, $[\varphi]$ and $[\psi]$ are unions of elements of Π_i, then $[\neg\varphi]$ and $[\varphi \cup \psi]$ have also this property, and similarly for the rest of the connectors. Thus this property holds for all the descriptions of i's knowledge. The proof for knowledge is similar.

3. Interpersonal independence

Our main results concern the relationship between the knowledge or belief of different agents. We show that knowledge is interpersonal independent in the sense that a description of the knowledge of one agent cannot serve as a description of the knowledge of another agent. Formally, we claim that, for $j \neq i$, a sentence in $\mathcal{S}_i^K$ cannot be logically equivalent to a sentence in $\mathcal{S}_j^K$. This claim requires fine tuning because sentences that are vacuous descriptions of knowledge and belief should be excluded. A sentence is a vacuous description of knowledge or belief if it is either a theorem, or a contradiction. Thus, if we take sentences φ in $\mathcal{S}_i^K$ and ψ in $\mathcal{S}_j^K$ that are theorems, then, of course, they are logically equivalent. This is also the case when we take two such sentences that are contradictory. Thus, interpersonal independence of knowledge claims that if a sentence in $\mathcal{S}_i^K$ *is* logically equivalent to a sentence in $\mathcal{S}_j^K$, for $i \neq j$, then these sentences are vacuous descriptions of knowledge. We show likewise that belief is also interpersonal independent. In what follows we assume for simplicity that $I = \{1, 2\}$.

Theorem 1. (Interpersonal independence of knowledge)
If $\varphi^1 \in \mathcal{S}_1^K$, $\varphi^2 \in \mathcal{S}_2^K$ and $\varphi^1 \Leftrightarrow \varphi^2$, then either both sentences are theorems, or both are contradictions.

The same result holds also for belief.

Theorem 2. (Interpersonal independence of belief)
If $\varphi^1 \in \mathcal{S}_1^B$, $\varphi^2 \in \mathcal{S}_2^B$ and $\varphi^1 \Leftrightarrow \varphi^2$, then either both sentences are theorems, or both are contradictions.

Interestingly, the proofs of the two theorems are fundamentally distinct. The proof of Theorem 2 relies on a stronger condition of interpersonal independence, which is satisfied by belief but not by knowledge. Specifically, this stronger form of independence requires not only that two agents' belief descriptions cannot be logically equivalent, but even that one cannot logically imply the other. Since knowledge does not satisfy strong interpersonal independence, the proof of Theorem 1 requires a different approach which we describe later. But first, we formally state that belief satisfies strong interpersonal independence.

Theorem 3. (Strong interpersonal independence of belief)
If $\varphi^1 \in \mathcal{S}_1^B$, $\varphi^2 \in \mathcal{S}_2^B$ and $\varphi^1 \Rightarrow \varphi^2$ then either φ^1 is a contradiction or φ^2 is a theorem.

Theorem 3 easily implies Theorem 2. If $\varphi^1 \in \mathcal{S}_1^B$, $\varphi^2 \in \mathcal{S}_2^B$ and $\varphi^1 \Leftrightarrow \varphi^2$, then $\varphi^1 \Rightarrow \varphi^2$. Thus, by Theorem 3 either φ^1 is a contradiction, and therefore, as $\varphi^1 \Leftrightarrow \varphi^2$ it follows that φ^2 is also a contradiction, or else, φ^2 is a theorem, and then again, φ^1 is a theorem too.

Obviously, knowledge does not have strong interpersonal independence. The culprit is the truth axiom. Consider, for instance, the two sentences, $K_1 p$ and $\neg K_2 \neg p$ for some atomic sentence p. Obviously, the first sentence is not a contradiction and the second is not a theorem. Yet, by the truth axiom, $K_1 p \Rightarrow p$, and also $p \Rightarrow \neg K_2 \neg p$. Thus, by the transitivity of logical implication, $K_1 p \Rightarrow \neg K_2 \neg p$. Hence, a sentence that describes 1's knowledge implies a certain sentence that describes 2's knowledge. Thus, strong interpersonal independence does not hold for knowledge.

In the next section, we provide a short proof of Theorem 3. In the section that follows we give the more elaborate proof of Theorem 1.

4. Strong interpersonal independence of belief

To prove Theorem 3 we use the following proposition which is a special case of the theorem.

Proposition 1. *If $B_1\varphi^1$ and $B_2\varphi^2$ are consistent sentences in $\mathcal{S}^B$, then $B_1\varphi^1 \wedge B_2\varphi^2$ is consistent.*

Proof: If $B_1\varphi^1$ and $B_2\varphi^2$ are consistent, then for $k = 1,2$ there is a model $M^k = (\Omega^k, (\rightsquigarrow_i^k)_{i=1,2}, [\,\cdot\,]^k)$ and a state $\omega^k \in \Omega^k$, such that $\omega^k \in [B_k\varphi^k]_{M^k}$. We can assume that $\Omega_1 \cap \Omega_2 = \emptyset$.

We construct a model $M = (\Omega, (\rightsquigarrow_i)_{i=1,2}, [\,\cdot\,])$ by taking the union of the models M^1 and M^2 and adding a state ω_0. We will show that in M, $\omega_0 \in B_1\varphi^1 \wedge B_2\varphi^2$, which proves the consistency of $B_1\varphi^1 \wedge B_2\varphi^2$.

We set $\Omega = \{\omega_0\} \cup \Omega^1 \cup \Omega^2$, where $\omega_0 \notin \Omega^1 \cup \Omega^2$. The restriction of $(\rightsquigarrow_i)$ for $i = 1,2$ to Ω^k, $k = 1,2$ is $\rightsquigarrow_i^k$. For ω_0 we set $\omega_0 \rightsquigarrow_1 \omega$ when $\omega \in \Sigma_1^1(\omega^1)$ – the support at ω^1 in the model M^1, and $\omega_0 \rightsquigarrow_2 \omega$ when $\omega \in \Sigma_2^2(\omega^2)$. Finally, we set $[\,\cdot\,] = [\,\cdot\,]^1 \cup [\,\cdot\,]^2$. It is easy to see that $\rightsquigarrow_i$, for $i = 1,2$ is serial, transitive, and Euclidean, and thus M is a model of belief.

We now show that for any sentence φ, $[\varphi] \cap (\Omega^1 \cup \Omega^2) = [\varphi]_{M^1} \cup [\varphi]_{M^2}$. For $\varphi \in A$ this follows from the definition of $[\,\cdot\,]$. Suppose that φ and ψ satisfy this equality. Then, $[\varphi \cup \psi] \cap (\Omega^1 \cup \Omega^2) = ([\varphi] \cap (\Omega^1 \cup \Omega^2)) \cup ([\eta] \cap (\Omega^1 \cup \Omega^2)) = ([\varphi]_{M^1} \cup [\varphi]_{M^2}) \cup ([\psi]_{M^1} \cup [\psi]_{M^2}) = [\varphi \cup \psi]_{M^1} \cup [\varphi \cup \psi]_{M^2}$. Also, $[\neg\varphi] \cap (\Omega^1 \cup \Omega^2) = (\Omega^1 \cup \Omega^2) \setminus [\varphi] = (\Omega^1 \setminus [\varphi]_{M_1}) \cup (\Omega^2 \setminus [\varphi]_{M_2}) = [\neg\varphi]_{M_1} \cup [\neg\varphi]_{M_2}$. Finally, as for $i = 1,2$, $\rightsquigarrow_i$ coincides with $\rightsquigarrow_i^1$ on Ω^1 and with $\rightsquigarrow_i^2$ on Ω^2, it follows that $[B_i\varphi] \cap (\Omega^1 \cup \Omega^2) = ([B_i\varphi] \cap \Omega^1) \cup ([B_i\varphi] \cap \Omega^2)$ for $i = 1,2$.

Since $\omega^1 \in [B_1\varphi^1]_{M^1}$ and $\omega^2 \in [B_2\varphi^2]_{M^2}$ it follows that $\omega^1 \in [B_1\varphi^1]$ and $\omega^2 \in [B_2\varphi^2]$. Thus, $\Sigma_1(\omega^1) \subseteq [\varphi^1]$ and $\Sigma_2(\omega^2) \subseteq [\varphi^2]$. As $\Sigma_1(\omega_0) = \Sigma_1(\omega^1)$ and $\Sigma_2(\omega_0) = \Sigma_2(\omega^2)$ we conclude that $\omega_0 \in [B_1\varphi^1]$ and $\omega_0 \in [B_2\varphi^2]$. Hence, $\omega_0 \in [B_1\varphi^1 \wedge B_2\varphi^2]$. ∎

We need also a simple claim that helps to extend Proposition 1 to all sentences in $\mathcal{S}_i^B$.

Claim 2. *If $\varphi \in \mathcal{S}_i^B$, then $\varphi \Leftrightarrow B_i(\varphi)$. If $\varphi \in \mathcal{S}_i^K$, then $\varphi \Leftrightarrow K_i(\varphi)$*

Proof: By Claim 1, if $\varphi \in \mathcal{S}_i^B$, then in each model, $[\varphi]$ is a union of elements of Π_i. By definition, if $[\varphi]$ is a union of elements of Π_i then $[\varphi] = [B_i\varphi]$. Since this equality holds in any model, it follows that $\varphi \Leftrightarrow B_i(\varphi)$. The proof for knowledge is similar. ∎

Proof of Theorem 3: If φ^1 is not a contradiction and φ^2 is not a theorem, then φ^1 and $\neg\varphi^2$ are consistent. Since $\varphi^1 \in \mathcal{S}_1^B$ and $\neg\varphi^2 \in \mathcal{S}_2^B$, it follows by Claim 2 that $\varphi^1 \Leftrightarrow B_1\varphi^1$ and $\neg\varphi^2 \Leftrightarrow B_2\neg\varphi^2$. Therefore, $B_1\varphi^1$ and $B_2\neg\varphi^2$ are consistent. Hence, by Proposition 1, $B_1\varphi^1 \wedge B_2\neg\varphi^2$ is consistent. This shows, again by Claim 2, that $\varphi^1 \wedge \neg\varphi^2$ is consistent and thus φ^1 does not logically imply φ^2. ∎

5. Interpersonal independence of knowledge

The proof of the interpersonal independence of belief cannot work for knowledge, as knowledge does not have the property of strong interpersonal independence. We prove Theorem 1 using the following result, which involves only one sentence rather than two sentences as in Theorem 1.

Theorem 4. *If $\varphi \Rightarrow K_1(\varphi)$ and $\varphi \Rightarrow K_2(\varphi)$, then φ is either a contradiction or a theorem.*

Proof of Theorem 1: Assume that (a) $\varphi^1 \in \mathcal{S}_1^K$, (b) $\varphi^2 \in \mathcal{S}_2^K$, and (c) $\varphi^1 \Leftrightarrow \varphi^2$. Then from (a) and (b) we conclude, by Claim 2, that *(d)* $\varphi^1 \Leftrightarrow K_1\varphi^1$ and *(e)* $\varphi^2 \Leftrightarrow K_2\varphi^2$. By (c) and the definition of interpretation, it follows that (f) $K_2\varphi^1 \Leftrightarrow K_2\varphi^2$. Using (c), (e), and (f), we infer (g) $\varphi^1 \Leftrightarrow K_2\varphi^1$. Applying Theorem 4 to φ^1, using (d) and (g), we conclude that φ^1 is either a contradiction, and thus by (c), φ^2 is also a contradiction, or φ^1 is a theorem, and therefore by (c), φ^2 is also a theorem. ∎

We rephrase Theorem 4 in terms of common knowledge. Following Aumann (1976), we define in a partition model the *common knowledge partition* Π to be the meet of the partitions Π_1 and Π_2. That is, Π is the finest partition among the partitions that are coarser than both Π_1 and Π_2. Alternatively, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the fields generated by the elements of Π_1 and Π_2, respectively. Then $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a field and it is generated by the meet partition Π. Thus, each element of $\mathcal{F}$ is a union of elements of each of the partitions Π_1 and Π_2. As Π is included in $\mathcal{F}$, this statement applies also to elements in Π. We call an element of $\mathcal{F}$, a *common knowledge event.*[6] If a non-empty set $\Omega' \subseteq \Omega$ is a common knowledge event, then $M' = (\Omega', (\Pi'_i)_{i\in I}, [\,\cdot\,]')$ is a knowledge model, where Π'_i is the set of elements in Π_i that are contained in Ω', and $[\,\cdot\,]' = [\,\cdot\,] \cap \Omega'$.

Common knowledge events can be described also in terms of the accessibility relations, as follows. A *path of length $n \geq 0$ from ω to ω'* is a sequence $(\omega_k)_{k=0}^n$, such that $\omega_0 = \omega$, $\omega_n = \omega'$, and for each $k < n$, $\omega_k \rightsquigarrow_i \omega_{k+1}$ for some i. Denote by $\rightsquigarrow$ the transitive closure of the accessibility relations $\rightsquigarrow_1$ and $\rightsquigarrow_2$. That is, $\omega \rightsquigarrow \omega'$ when there is a path from ω to ω'. The relation $\rightsquigarrow$ is reflexive, symmetric, and transitive, i.e., it is an equivalence relation. We call $\rightsquigarrow$ the *common knowledge accessibility relation* for the following reason.

[6] Thus, the empty set is considered also as a common knowledge event.

Claim 3. *The partition of Ω into the equivalence classes of $\rightsquigarrow$ is the common knowledge partition, that is the meet partition.*

Proof: An event E is a union of equivalence classes of $\rightsquigarrow$ if and only if E is closed under $\rightsquigarrow$. That is, if $\omega \in E$ and $\omega \rightsquigarrow \omega'$, then $\omega' \in E$. This condition holds if and only if E is closed with respect to both $\rightsquigarrow_1$ and $\rightsquigarrow_2$. This latter condition holds if and only if for each $\omega \in E$ and i, $\Pi_i(\omega) \subseteq E$, which means that E is a union of elements of the meet. ∎

Note that the accessibility relation defines a directed graph where the set of vertices is Ω and (ω, ω') is an edge if $\omega \rightsquigarrow \omega'$. The meet is the partition of Ω into the maximally connected subsets of vertices of this graph. We therefore say that the model is *connected* when the meet is $\{\Omega\}$.

Claim 4. *The logical implications $\varphi \Rightarrow K_1\varphi$ and $\varphi \Rightarrow K_2\varphi$ hold if and only if $[\varphi]$ is a common knowledge event in each model.*

Proof: Note that by the truth axioms, the two implications hold if and only if $\varphi \Leftrightarrow K_1\varphi$ and $\varphi \Leftrightarrow K_2\varphi$. By the definition of the interpretation of sentences $K_i\psi$, $[K_1\varphi]$ is a union of elements of Π_1 and $[K_2\varphi]$ is a union of elements of Π_2 in each model. If $\varphi \Leftrightarrow K_1\varphi$ and $\varphi \Leftrightarrow K_2\varphi$, then $[\varphi]$ is a union of elements of each of the partitions Π_1 and Π_2 in each model, and hence it is a common knowledge event in every model. Conversely, if $[\varphi]$ is a union of elements of each of the partitions Π_1 and Π_2, then $[\varphi] = [K_1\varphi] = [K_1\varphi]$. If this holds for every model, then $\varphi \Leftrightarrow K_1\varphi$ and $\varphi \Leftrightarrow K_2\varphi$. ∎

We rephrase Theorem 4 in terms of common knowledge. First, we write the theorem in contrapositive form. The antecedent is described in the semantic conditions for a sentence to be a contradiction or a theorem. The consequence is replaced by its equivalent form in Claim 4.

Theorem 4. *If there is a model in which $[\varphi] \neq \emptyset$ and a model in which $[\neg\varphi] \neq \emptyset$, then there exists a model in which $[\varphi]$ is not a common knowledge event.*

We outline the proof plan.

(1) We use $\rightsquigarrow$ to define a metric on the state space of a model.
(2) It is shown that the truth of a sentence at a given state ω in a model depends only on the set of states close to ω, in terms of the metric.
(3) We introduce a technique of gluing two models at two elements of the partitions of the models to create a bigger model.
(4) By gluing a model to its isomorphic image, we show that if φ is true in some state in a model, then there exists a bigger, in terms of the metric, connected model, where φ is true at some state of the model.
(5) Using (3) we take two big connected models M_1 and M_2, where φ is true in some state in ω_1 in M_1 and $\neg\varphi$ is true in some ω_2 in M_2.

(6) We glue M_1 and M_2 at some partition elements which are far from ω_1 and ω_2. By (2), φ is still true at ω_1 and $\neg\varphi$ is true at ω_2. The resulting glued model M is connected. In M, $[\varphi]$ is a non-empty proper subset of the state space, and since the state space is the only common knowledge event in M, $[\varphi]$ is not a common knowledge event.

The metric. We define a metric d on Ω by letting $d(\omega, \omega')$ be the length of the shortest path from ω to ω'. That is, $d(\omega, \omega') = n$ if there is a path of length n from ω to ω' and the length of any other path from ω to ω' is at least n; and $d(\omega, \omega') = \infty$ if there is no path connecting ω to ω'. The *ball* of radius n around ω is defined as $B(\omega, n) = \{\omega' \mid d(\omega, \omega') \leq n\}$.

Claim 5. *For $n \geq 1$, $B(\omega, n) = \bigcup_{i=1,2} \bigcup_{\omega' \in B(\omega, n-1)} \Pi_i(\omega')$.*

Proof: Suppose that $\omega' \in B(\omega, n-1)$ and $\bar{\omega} \in \Pi_i(\omega')$. Then, $d(\bar{\omega}, \omega) \leq d(\bar{\omega}, \omega') + d(\omega', \omega) \leq 1 + (n-1) = n$ and hence $\bar{\omega} \in B(\omega, n)$. Conversely, if $\bar{\omega} \in B(\omega, n)$, then there exist i and ω' such that $\omega' \rightsquigarrow_i \bar{\omega}$ and $d(\omega, \omega') \leq n-1$. Thus, for a world $\omega' \in B(\omega, n-1)$, $\bar{\omega} \in \Pi_i(\omega')$. ∎

We now formalize the idea expressed in point (2) of the proof plan by showing that the truth of a sentence φ at a world ω depends only on a certain ball centered at ω. First, we define the restriction of a model to a ball. For a state ω in model $M = (\Omega, (\Pi_i)_{i \in I}, [\cdot])$ and $n \geq 0$ we define *the restriction of M to the ball* $B(\omega, n)$ to be the model $M(\omega, n) := (B(\omega, n), (\Pi'_i)_{i \in I}, [\cdot]')$, where $B(\omega, n)$ is the set of states; for each state $\omega' \in B(\omega, n)$, $\Pi'_i(\omega') = \Pi_i(\omega') \cap B(\omega, n)$; and for each $p \in A$, $[p]' = [p] \cap B(\omega, n)$.

The depth of a sentence is defined recursively as follows: for an atomic sentence p, $\text{depth}(p) = 0$; for a negation $\neg\varphi$, $\text{depth}(\neg\varphi) = \text{depth}(\varphi)$; for a conjunction $\varphi \wedge \varphi'$, $\text{depth}(\varphi \wedge \varphi') = \max(\text{depth}(\varphi), \text{depth}(\varphi'))$; and for the sentence $K_i(\varphi)$, $\text{depth}(K_i(\varphi)) = \text{depth}(\varphi) + 1$. It turns out that the truth of a sentence φ at a given world ω in a model M is determined by its truth at the same ω in the model $M(\omega, n)$, where n is the depth of φ.

We state this formally in the following proposition.[7]

Proposition 2. *Let φ be a sentence of depth n. Then $M, \omega \models \varphi$ if and only if $M(\omega, n), \omega \models \varphi$.*[8]

Proof: We prove by induction on n. For $n = 0$, $B(\omega, 0) = \{\omega\}$. By the definition of $M(\omega, 0)$, for any atomic sentence p, $M, \omega \models p$ if and only if $M(\omega, 0), \omega \models p$. Since any sentence φ of depth 0 is generated by atomic sentences using propositional connectors, the proposition holds if and only if $M(\omega, 0), \omega \models \varphi$.

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[7]A similar result for logics with one modality only is stated in Nguyen (2000).

[8]As defined above, $M, \omega \models \varphi$ means that φ is true at ω in the model M.

Suppose that we proved the claim for $n-1 \geq 0$. Let φ be of depth n and fix a state ω_0 in the model M. We need to show that

$$M, \omega_0 \models \varphi \text{ if and only if } M(\omega_0, n), \omega_0 \models \varphi. \tag{1}$$

To evaluate the truth of φ at ω_0 in M we need to know the truth at ω_0 of atomic sentences and sentences of the form $K_i\psi$ for sentences ψ of degree $n-1$. For this we need to know the truth of such ψ's in states in $\Pi_i(\omega_0)$. Similarly, for the truth of φ at ω_0 in the model $M(\omega_0, n)$ with $\Pi'_i(\omega_0)$ instead of $\Pi_i(\omega_0)$. But by Claim 5, for $i = 1, 2$, $\Pi_i(\omega_0) = \Pi'_i(\omega_0)$. Thus, it is enough to show that for $i = 1, 2$, for any state $\omega \in \Pi_i(\omega_0)$ and ψ of depth $n-1$,

$$M, \omega \models \psi \text{ if and only if } M(\omega_0, n), \omega \models \psi. \tag{2}$$

By the induction hypothesis,

$$M, \omega \models \psi \text{ if and only if } M(\omega, n-1), \omega \models \psi. \tag{3}$$

Thus, it is enough to show that

$$M(\omega_0, n), \omega \models \psi \text{ if and only if } M(\omega, n-1), \omega \models \psi. \tag{4}$$

Since $d(\omega_0, \omega) \leq 1$, it follows that $B(\omega, n-1) \subseteq B(\omega_0, n)$. Let $M(\omega_0, n)(\omega, n-1) = (B(\omega, n-1), (\Pi''_i)_{i\in I}, [\cdot]'')$ be the restriction of $M(\omega_0, n)$ to $M(\omega, n-1)$.

Again, by the induction hypothesis,

$$M(\omega_0, n), \omega \models \psi \text{ if and only if } M(\omega_0, n)(\omega, n-1), \omega \models \psi. \tag{5}$$

We complete the proof by showing that the RHS of (4) holds if and only if the RHS of (5) holds. We do it by showing that $M(\omega_0, n)(\omega, n-1) = M(\omega, n-1)$. Let $M(\omega, n-1) = (B(\omega, n-1), (\Pi'''_i)_{i\in I}, [\cdot]''')$.[9] For every $\omega' \in B(\omega, n-1)$, $\Pi'''_i(\omega') = \Pi_i(\omega') \cap B(\omega, n-1) = \Pi_i(\omega') \cap B(\omega_0, n) \cap B(\omega, n-1) = \Pi'_i(\omega') \cap B(\omega, n-1) = \Pi''_i(\omega')$. The proof that $[p]''' = [p]''$ for any atomic sentence p, is similar. ∎

Constructing big connected models. In the sequel, we create a new model M by combining two disjoint models, M^1 and M^2, in a process called *gluing*. Here is how it works.

Let $M^1 = (\Omega^1, (\Pi^1_i)_{i\in I}, [\cdot]^1)$ and $M^2 = (\Omega^2, (\Pi^2_i)_{i\in I}, [\cdot]^2)$ be two models with $\Omega^1 \cap \Omega^2 = \emptyset$. Fix i and let $\pi^1 \in \Pi^1_i$ and $\pi^2 \in \Pi^2_i$. The *gluing* of M^1 and M^2 at π_1 and π_2 is the model $M = (\Omega, (\Pi_i)_{i\in I}, [\cdot])$ defined as follows: $\Omega = \Omega^1 \cup \Omega^2$; $[\cdot] = [\cdot]^1 \cup [\cdot]^2$; for $j \neq i$, $\Pi_j = \Pi^1_j \cup \Pi^2_j$: and finally, $\Pi_i = (\Pi^1_i \setminus \{\pi^1\}) \cup (\Pi^2_i \setminus \{\pi^2\}) \cup \{\pi^1 \cup \pi^2\}$.

In the following proposition, we state and prove formally that the shortest path in M from $\omega^1 \in \Omega^1$ to $\omega^2 \in \Omega^2$ crosses $\pi^1 \cup \pi^2$ only once.

[9] To avoid confusion, we recall that we use Π, Π', Π'' and Π''' to denote the partitions in the models $M, M(\omega_0, n), M(\omega_0, n)(\omega, n-1)$ and $M(\omega, n-1)$, respectively.

Proposition 3. *Let $\omega^1 \in \Omega^1$, $\omega^2 \in \Omega^2$, and let $(\omega_k)_{k=0}^n$ be a shortest path from ω^1 to ω^2 in the model M. Then, there exists ℓ such that $(\omega_k)_{k=0}^{\ell} \subseteq \Omega^1$ and $(\omega_k)_{k=\ell+1}^n \subseteq \Omega^2$, and thus $\omega_\ell \in \pi^1$ and $\omega_{\ell+1} \in \pi^2$.*

Proof: By definition, $\omega_0 = \omega^1 \in \Omega^1$. Thus, ℓ, the largest index such that $(\omega_k)_{k=0}^{\ell} \subseteq \Omega^1$ is well defined. Similarly, $\omega_n = \omega^2 \in \Omega^2$. Thus, m, the smallest index such that $(\omega_k)_{k=m}^n \subseteq \Omega^2$ is well defined. Obviously, $m > \ell$. By definition, $\omega_{\ell+1} \in \Omega^2$. Thus, $\omega_\ell \in \pi^1$ and $\omega_{\ell+1} \in \pi^2$. Similarly, $\omega_{m-1} \in \Omega^1$ and therefore $\omega_{m-1} \in \pi^1$ and $\omega_m \in \pi^2$. It is impossible that $m > \ell + 1$, because then, the segment of the path, which starts with ω_ℓ and ends at Ω_m contains at least three worlds, and can be replaced by (ω_ℓ, ω_m) which contains only two worlds, and thus contradicts the assumption that n is the length of a shortest path from ω^1 to ω^2. Consequently, $\omega_m = \omega_{\ell+1}$. ∎

Next, we consider the gluing of two disjoint models that have the same structure. Formally, the models M^1 and M^2 are *isomorphic* if there is a bijection $f : \Omega^1 \to \Omega^2$ that preserves the partitions and the assignment of parameters to state. That is, for any ω, $f(\Pi_i^1(\omega)) = \Pi_i^2(f(\omega))$, and for any $p \in A$, $f([p]^1) = [p]^2$. The isomorphism of the two glued models guarantees that the truth of a sentence in one of the models is preserved in the glued model, as stated in the following proposition.

Proposition 4. *Let M be the gluing of two isomorphic models M^1 and M^2, with a bijection f, at $\pi \in \Pi_i$ and $f(\pi)$. Then for any φ, $[\varphi] = [\varphi]^1 \cup [\varphi]^2$. In other words, φ is true in M at a state ω if and only if it is true at ω in the model M^j where $\omega \in \Omega^j$.*

Proof: The claim holds for atomic sentences by definition of $[\,\cdot\,]$. If it holds for φ then $[\neg\varphi] = \neg[\varphi] = \neg[\varphi]^1 \cap \neg[\varphi]^2 = (\Omega_2 \cup (\Omega_1 \setminus [\varphi]^1)) \cap (\Omega_1 \cup (\Omega_2 \setminus [\varphi]^2)) = (\Omega_1 \setminus [\varphi]^1) \cup (\Omega_2 \setminus [\varphi]^2) = [\neg\varphi]^1 \cup [\neg\varphi^2]$. If the claim holds for φ and η, then $[\varphi \cup \eta] = [\varphi] \cup [\eta] = ([\varphi]^1 \cup [\varphi]^2) \cup ([\eta]^1 \cup [\eta]^2) = [\varphi \cup \eta]^1 \cup [\varphi \cup \eta]^2$. Suppose the claim holds for φ and consider $j \neq i$. Then $[K_j\varphi]$ is the union of all the sets in Π_j included in $[\varphi]$. Since $\Pi_j = \Pi_j^1 \cup \Pi_j^2$, and since $[\varphi] = [\varphi]^1 \cup [\varphi]^2$, it follows that this union is the union of the elements of Π_j^1 contained in $[\varphi]^1$ and the elements of Π_j^2 contained in $[\varphi]^2$. Thus, $[K_j\varphi] = [K_j\varphi]^1 \cup [K_j\varphi]^2$. Next, $[K_i\varphi]$ is a union of the sets in Π_i that are contained in $[\varphi]$. If $\pi \cup f(\pi)$ is not contained in $[\varphi]$ then $[K_i\varphi] = [K_i\varphi]^1 \cup [K_i\varphi]^2$, as in the case of $K_j\varphi$. If, otherwise, $\pi \cup f(\pi)$ is contained in $[\varphi]$, then π is contained in $[\varphi]^1$ and $f(\pi)$ is contained in $[\varphi]^2$. This implies again that $[K_i\varphi] = [K_i\varphi]^1 \cup [K_i\varphi]^2$. ∎

We now prove the following proposition for the gluing of two isomorphic models.

Proposition 5. *Let φ be a sentence that is not a contradiction. Then for each $n \geq 0$, there exists a connected model M with states ω and ω' such that $\omega \in [\varphi]$, and $d(\omega, \omega') > n$.*

Proof: Let M^1 be a model with a state ω such that $\omega \in [\varphi]$. Since $\Pi^1(\omega)$ can be considered as a model in which φ is true at ω, we assume, without loss of generality, that $\Pi^1 = \{\Omega^1\}$. If $B(\omega, n) \neq \Omega^1$ for all n, we are done. Otherwise, let $B(\omega, n) = \Omega^1$ for some n. Let m be the least number for which this equality holds. We show that we can find a model M such that $B(\omega, m) \neq \Omega$.

Assume first that $m \geq 2$. By the definition of m, there exists a state $\bar{\omega} \in \Omega^1$ such that $d(\omega, \bar{\omega}) = m$. Let $\pi = \Pi_i(\bar{\omega})$ for some i. Let M^2 be a model isomorphic to M^1 by bijection f, and M be a gluing of M^1 and M^2 at π and $f(\pi)$. By Proposition 4, in the model M, φ is true at ω.

We show that in M, $d(\omega, f(\omega)) > m$. Let $(\omega_k)_{k=0}^n$ be the shortest path from ω to $f(\omega)$. By proposition 3, there exists ℓ such that $\omega_\ell \in \pi$ and $\omega_{\ell+1} \in f(\pi)$. Since d is a metric, $d(\omega, \omega_\ell) \geq d(\omega, \bar{\omega}) - d(\omega_\ell, \bar{\omega}) = m - 1$. Symmetrically, $d(f(\omega), \omega_{\ell+1}) \geq d(f(\omega), f(\bar{\omega})) - d(\omega_{\ell+1}), f(\bar{\omega})) = m - 1$. Thus, $d(\omega, f(\omega)) \geq d(\omega, \omega_\ell) + d(\omega_\ell, \omega_{\ell+1}) + d(\omega_{\ell+1}, f(\omega)) = (m-1) + 1 + (m-1) = 2m - 1 > m$.

Obviously, this proof does not work for $m = 0$ and $m = 1$. For $m = 0$, $B(\omega, 0) = \Omega^1$ and thus $\Omega^1 = \{\omega\}$. We define M to be the gluing of M^1 and an isomorphic model M^2 at, say, $\pi = \Pi_1(\omega)$ and $f(\pi)$. Clearly, in M, $B(\omega, 0) = \{\omega\} \neq \Omega$. For $m = 1$, $B(\omega, 1) = \Omega^1$, which means that $\Pi_i(\omega) = \Omega^1$ for at least one agent i. Suppose that $\Pi_1(\omega) = \Omega^1$, but $\Pi_2(\omega) \neq \Omega^1$. Then, there exists $\bar{\omega}$ such that $\omega \notin \Pi_2(\bar{\omega})$. Take a model M^2 isomorphic under f to M^1 and let M be the gluing of M^1 and M^2 at $\Pi_2(\bar{\omega})$ and $f(\Pi_2(\bar{\omega}))$. The shortest path from ω to $f(\omega)$ must include two worlds, $\omega' \in \Pi_2(\bar{\omega})$ and $\omega'' \in f(\Pi_2(\bar{\omega}))$. Since $\omega \neq \omega'$, it follows that $d(\omega, \omega') = 1$. Symmetrically, $d(f(\omega), \omega'') = 1$. Thus, $d(\omega, f(\omega)) = 3$.

If $\Pi_1(\omega) = \Pi_2(\omega) = \Omega^1$, we glue Ω^1 with an isomorphic Ω^2 at $\Pi_1(\omega)$ and $f(\Pi_1(\omega))$. In this model still $B(\omega, 1) = \Omega$, but $\Pi_2(\omega) \neq \Omega$, and this is the case we dealt with above.

Note that since M^1 and M^2 are connected, so is M. ∎

Proof of Theorem 4: Let φ be a sentence of depth n. Suppose that there exists a model $M^1 = (\Omega^1, (\Pi_i^1)_{i \in I}, [\,\cdot\,]^1)$ and $\omega^1 \in \Omega^1$ such that $M^1, \omega^1 \models \varphi$. Suppose further, that there exists a model $M^2 = (\Omega^2, (\Pi_i^2)_{i \in I}, [\,\cdot\,]^2)$ and $\omega^2 \in \Omega^2$ such that $M^2, \omega^2 \models \neg\varphi$. By Proposition 5, we can assume that M^1 and M^2 are connected, and for $i = 1, 2$ there is a state $\hat{\omega}^i \in \Omega^i$, such that $d^i(\omega^i, \hat{\omega}^i) \geq n + 2$, where d^i is the metric in the model M^i. We can further assume that $\Omega^1 \cap \Omega^2 = \emptyset$.

Let M be the model which is the gluing of M^1 and M^2 at $\pi^1 = \Pi_i^1(\hat{\omega}^1)$ and $\pi^2 = \Pi_i^2(\hat{\omega}^2)$. Since M^1 and M^2 are connected, M is also connected. We show now that in M, $[\varphi]$ is not a common knowledge event, completing the proof of the theorem.

We denote by B^i balls in the model M^i. For any $\omega' \in \pi^1$, $d^1(\omega^1, \omega') \geq d^1(\omega^1, \hat{\omega}^1) - d^1(\omega', \hat{\omega}^1) \geq (n+2) - 1 = n + 1$. If $\omega \in B^1(\omega^1, n-1)$ and $\omega' \in \pi^1$, then $d^1(\omega', \omega) \geq d^1(\omega', \omega^1) - d^1(\omega, \omega^1) \geq (n+1) - (n-1) = 2$. Since this is true for every $\omega' \in \pi^1$, it follows that $\Pi_i^1(\omega) \neq \pi^1$ for every $\omega \in B^1(\omega^1, n-1)$.

By the definition of the partitions Π_i in M and Claim 5, $B(\omega^1, n) = B^1(\omega^1, n)$. Furthermore, $M(\omega^1, n)$, the restriction of M to $B(\omega^1, n)$, and $M^1(\omega^1, n)$, the restriction of M^1 to $B^1(\omega^1, n)$, are the same model. As $M^1, \omega^1 \models \varphi$, Proposition 2 implies that $M^1(\omega^1, n), \omega^1 \models \varphi$, and thus $M(\omega^1, n), \omega^1 \models \varphi$. By the same proposition, $M, \omega^1 \models \varphi$. By similar arguments we conclude that $M, \omega^2 \models \neg\varphi$. Therefore, $\emptyset \subsetneq [\varphi] \subsetneq \Omega$ in M. Since M is connected, Ω does not contain a proper subset which is common knowledge, implying that $[\varphi]$ is not a common knowledge event in M, as desired. ∎

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